3719. Replacement. Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.

Prove that if $a,b,c>0$, then

$$\frac{a}{\sqrt{b^2+\frac{1}{4}bc+c^2}}+\frac{b}{\sqrt{c^2+\frac{1}{4}ca+a^2}}+\frac{c}{\sqrt{a^2+\frac{1}{4}ab+b^2}}\geq 2.$$

Solution by Arkady Alt , San Jose ,California, USA.

Note that $\sum_{cyc}\frac{a}{\sqrt{b^2+\frac{1}{4}bc+c^2}}\geq 2\Leftrightarrow\sum_{cyc}\frac{a}{\sqrt{4b^2+bc+4c^2}}\geq 1.$

Since function $t\mapsto\frac{1}{\sqrt{t}}:(0,\infty)\to(0,\infty)$ is concave up, then assuming $a+b+c=1$ (due homogeneity) and applying Jensen's Inequality

$$a\cdot\frac{1}{\sqrt{x}}+b\cdot\frac{1}{\sqrt{y}}+c\cdot\frac{1}{\sqrt{z}}\geq\frac{1}{\sqrt{ax+by+cz}}$$

to $(x,y,z)=(4b^2+bc+4c^2,4c^2+ca+4a^2,4a^2+ab+4b^2)$ we obtain

$$\sum_{cyc}\frac{a}{\sqrt{4b^2+bc+4c^2}}\geq\frac{1}{\sqrt{\sum_{cyc}a(4b^2+bc+4c^2)}}=\frac{1}{\sqrt{4(ab+bc+ca)-9abc}}\geq 1$$

because $1\geq 4(ab+bc+ca)-9abc\Leftrightarrow\sum_{cyc}a(a-b)(a-c)\geq 0$ (Shure Inequality).

Solution 2.

Applying Holder Inequality $(x+y+z)^2(u+v+w)\geq\left(\sqrt[3]{x^2u}+\sqrt[3]{y^2v}+\sqrt[3]{z^2w}\right)^3$

to $(x,y,z)=\left(\frac{a}{\sqrt{4b^2+bc+4c^2}},\frac{b}{\sqrt{4c^2+ca+4a^2}},\frac{c}{\sqrt{4a^2+ab+4b^2}}\right)$ and

$(u,v,w)=(a(4b^2+bc+4c^2),b(4c^2+ca+4a^2),c(4a^2+ab+4b^2))$ we obtain

$$\left(\sum_{cyc}\frac{a}{\sqrt{4b^2+bc+4c^2}}\right)^2\cdot\sum_{cyc}a(4b^2+bc+4c^2)\geq(a+b+c)^3\Leftrightarrow$$

$$\left(\sum_{cyc}\frac{a}{\sqrt{4b^2+bc+4c^2}}\right)^2\geq\frac{(a+b+c)^3}{\sum_{cyc}a(4b^2+bc+4c^2)}=\frac{(a+b+c)^3}{4(a+b+c)(ab+bc+ca)-9abc}$$

where $\frac{(a+b+c)^3}{4(a+b+c)(ab+bc+ca)-9abc}\geq 1\Leftrightarrow$

$(a+b+c)^3\geq 4(a+b+c)(ab+bc+ca)-9abc\Leftrightarrow\sum_{cyc}a(a-b)(a-c)\geq 0.$